Math 53, Discussion 107

Friday, April 17, 2020 9:42 AM

16.5 (Curl and Divergence)

16.6 (Parametric Surfaces and Their Areas)

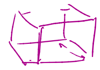
$\vec{a} \cdot \vec{b}$

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \partial_y F_3 - \partial_z F_2 \\ \vdots \\ \vdots \end{pmatrix}$$

$\langle \partial_x, \partial_y, \partial_z \rangle$ given a v.F $F: \mathbb{R}^3 \to \mathbb{R}^3$

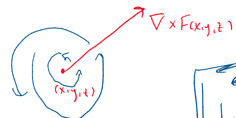
Curl $F = \nabla \times F = \langle \partial_x, \partial_y, \partial_z \rangle \times F$: $\mathbb{R}^3 \to \mathbb{R}^3$

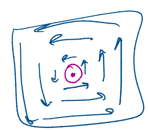
Div $F = \nabla \cdot F = \langle \partial_x, \partial_y, \partial_z \rangle \cdot F$: $\mathbb{R}^3 \to \mathbb{R}$



interpretation of curl

- F is velocity field of a fluid in $\mathbb{R}^3$
- @ (x,y,z), $\nabla \times F(x,y,z)$





Theorem:

1. $Curl(\nabla f) = 0$
2. $curl(F) = 0$ implies F is conservative (on a simply connected set) — set without holes
3. Div(Curl(F))=0

∇ | gradient: $\nabla f = \langle f_x, f_y, f_z \rangle$
divergence: $\nabla \cdot F = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$
Curl: $\nabla \times F$

Formula: $\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D div(\vec{F})\, dA$

let $\vec{F} = \nabla f$ RHS = $div(\nabla f) = 0$

$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_a^b \vec{F} \cdot \vec{n}(r(t))\, |r'(t)|\, dt$ $\quad r(t) = (x, y)$

$\vec{T}(t) = \dfrac{x'(t)}{|r'(t)|}\hat{\imath} + \dfrac{y'(t)}{|r'(t)|}\hat{\jmath}$

$\vec{n}(t) = \dfrac{y'(t)}{|r'(t)|}\hat{\imath} - \dfrac{x'(t)}{|r'(t)|}\hat{\jmath}$

$$\int_C F_1\, dx + F_2\, dy = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$\int_a^b \left(\frac{F_1 y'(t)}{|r'(t)|} - \frac{F_2 x'(t)}{|r'(t)|} \right) |r'(t)|\, dt = \int_a^b F_1 y'(t) - F_2 x'(t)\, dt$$

$$= \int_C F_1\, dy - F_2\, dx = \iint \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA$$

$$-\iint \nabla \cdot F\, dA$$

Exercises:

1. Compute curl and divergence of
   a. $xyz\boldsymbol{i} - x^2y\boldsymbol{k}$
   b. $\cos xz\, \boldsymbol{j} - \sin xy\, \boldsymbol{k}$
2. Determine whether or not the vector field is conservative, if it is, find its potential
   a. $\langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$
   b. $\langle 2xy, (x^2 + 2yz), y^2 \rangle$
   c. $\langle ye^{-x}, e^{-x}, 2z \rangle$
3. For $\vec{r} = \langle x, y, z \rangle$ verify the following
   a. $\nabla \cdot \vec{r} = 3$ $\quad (r = \|\vec{r}\|)$
   b. $\nabla \cdot (r\vec{r}) = 4r$
   c. $\nabla r = \dfrac{\vec{r}}{r}$
   d. $\nabla \times \vec{r} = 0$
4. Show that if f is harmonic, then $\oint_C \nabla f \cdot \vec{n}\, ds = 0$

$div(grad(f)) = 0$

$f_{xx} + f_{yy} + f_{zz} = 0$

given F, want to find f s.t. $\nabla f = F$

$\boxed{f_x = F_1 = ye^{-x}}$

- integrate wrt x $\Rightarrow$ $f = -ye^{-x} + g(y,z)$
- Now differentiate wrt y & compare to F_2

$-e^{-x} = F_2 = \partial_y f = \boxed{-e^{-x} + g_y(y,z)}$

$\Rightarrow g_y(y,z) = 0$ $\qquad \dfrac{\partial}{\partial y} g(y,z)$

integrate $\Rightarrow f = -ye^{-x} + g(z)$

differentiate wrt z

$\Rightarrow f_z = g'(z) = F_3 = 2z$

$g'(z) = 2z$

$\Rightarrow g = z^2 + c$

$f = -ye^{-x} + z^2 + c$

~~$f = \sin(xz) + y^2x + zy$~~ $\qquad$ F $\qquad \begin{pmatrix} z\cos(xz) + y^2 \\ 2yx + z \\ x\cos(xz) + y \end{pmatrix}$